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Inverse Eigenvalue Problem in Structural Design

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A method is presented for the design of a structure with specified low-order natural frequencies. This is based on the solution of a partial inverse eigenvalue problem. The method can also be used for design problems in which certain bands, devoid of any frequency, are specified. This can further be used to generate initial feasible designs for optimum design problems with frequency constraints. A few truss problems are discussed to illustrate the method. Though only truss examples are discussed, the approach is general and can be applied to any type of structure.

Introduction

THE design of a structure either with specified natural frequencies or with frequencies that lie outside certain specified bands is a problem that occurs frequently in structural engineering. The natural frequencies of a structure are usually computed using the stiffness and mass matrices. The stiffness matrix is obtained by the summation of element stiffness matrices, which are functions of certain parameters such as the areas of cross section of the members. The mass matrix also is obtained in a similar way. The problem here is to determine the set of parameters such that the associated eigenvalue problem has the specified eigenvalues. This is called an inverse eigenvalue problem.

An approach for the solution of this problem is to consider the eigenvalues themselves as functions of the parameters. A set of nonlinear equations is obtained by equating them to the specified eigenvalues. This set of nonlinear equations can be solved by the Newton-Raphson method or one of its variants.

Inverse eigenvalue problems corresponding to ordinary eigenvalue problems have been investigated[1-5] and a few methods have been suggested. In these problems, the number of specified eigenvalues and the number of unknown parameters to be determined are the same as the order of the matrix.

In structural design, our interest often lies only in few low-order eigenvalues, whereas the number of design parameters can be much more than the number of specified eigenvalues. Moreover, some side constraints for the design parameters will also be present. This leads to a partial inverse eigenvalue problem with constraints. This problem is usually solved by nonlinear optimization techniques or some scaling procedures as part of methods for the minimum weight design of structures under frequency constraints.[6-8]

Inverse problems in vibrations have been an area of interest on their own. This area is concerned with the determination or estimation of the material properties of a structure from its spectral data, i.e., the frequencies and/or the mode shapes. Issues such as existence, uniqueness, and reconstruction have been explored for a class of problems.[9-17] Reference 18 gives a survey, and Ref. 19 provides a detailed description of the methods used.

Our aim in this paper is to extend a method for the ordinary inverse eigenvalue problem to the partial generalized inverse eigenvalue problem and bring out its usefulness in structural design. In the following section, we briefly review the inverse eigenvalue problem and a method of solution. The extension of the method for use in structural design is discussed next. The application of the method is illustrated for a few truss design problems. One of the problems is used to demonstrate the interactive use of the method to design a structure whose frequencies lie outside certain specified bands.

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Inverse Eigenvalue Problem

In this section we define the inverse eigenvalue problem and briefly review a solution procedure based on the Newton-Raphson method. Reference 5 gives an excellent treatment of the topic.

Definition of the Problem

Let $A_0, A_1, A_2, \ldots, A_n$ be a set of $(n \times n)$ real symmetric matrices of constants, let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ be a vector of n parameters, and let $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$ be a set of n real constants arranged in ascending order. The inverse eigenvalue problem is to determine the vector $\boldsymbol{a}$ such that $\boldsymbol{\lambda}^*$ is the set of the eigenvalues of the matrix

$$A(\boldsymbol{a}) = A_0 + \sum_{i=1}^{n} a_i A_i \tag{1}$$

Method

Let $\lambda(\boldsymbol{a}) = [\lambda_1(\boldsymbol{a}), \lambda_2(\boldsymbol{a}), \ldots, \lambda_n(\boldsymbol{a})]$ be the eigenvalues of $A(\boldsymbol{a})$ arranged in ascending order. We get the following n nonlinear equations in the n unknowns $\boldsymbol{a}$:

$$\lambda_1(\boldsymbol{a}) = \lambda_1^*$$

$$\lambda_2(\boldsymbol{a}) = \lambda_2^*$$

$$\ldots$$

$$\lambda_n(\boldsymbol{a}) = \lambda_n^*$$

Let $\boldsymbol{a}^*$ be the solution. It can be obtained by solving this system of equations by Newton's method. Let the square matrix $\boldsymbol{J}$ of order n be the Jacobian of the functions on the left side of these equations. The (i,j)th element of the Jacobian is defined by

$$J_{ij} = \frac{\partial \lambda_i}{\partial a_j}$$

Let $\lambda(\boldsymbol{a}^k)$ be the set of the eigenvalues of $A(\boldsymbol{a})$ at the kth iterate $\boldsymbol{a} = \boldsymbol{a}^k$. Then the next approximation $\boldsymbol{a}^{k+1}$ to the solution $\boldsymbol{a}^*$ can be obtained from

$$\boldsymbol{J}(\boldsymbol{a}^{k+1} - \boldsymbol{a}^k) = \lambda(\boldsymbol{a}^k) \tag{2}$$

If the orthonormalized eigenvectors $\boldsymbol{x}_i(\boldsymbol{a}^k)$, $i = 1, n$ corresponding to the eigenvalues $\lambda_i(\boldsymbol{a}^k)$, are also available, J_{ij} can be written as

$$J_{ij} = \boldsymbol{x}_i^T(\boldsymbol{a}^k)\, A_j\, \boldsymbol{x}_i(\boldsymbol{a}^k) \tag{3}$$

We write Eq. (2) in a different form. Using Eq. (1) and the standard relation

$$x_i^T(a^k)\,A(a^k)\,x_i(a^k) = \lambda_i(a^k)$$

we have

$$\sum_{j=1}^{n} a_j^{k+1}\, x_i^T(a^k)\,A_j\,x_i(a^k) = \lambda_i(a^k) - x_i^T(a^k)\,A_0\,x_i(a^k)$$

Equivalently

$$J\,a^{k+1} = \lambda(a^k) - b$$

where

$$b_i = x_i^T(a^k)\,A_0\,x_i(a^k), \qquad i = 1, n \tag{4}$$

Now Eq. (2) becomes

$$J\,a^{k+1} = \lambda^* - b \tag{5}$$

Algorithm Steps

The steps of the algorithm for the inverse eigenvalue problem are as follows:

1) Choose an initial approximate solution a^0 and set the iteration index k to 0.
2) Form $A(a^k)$ and calculate its eigenvalues $\lambda(a^k)$ arranged in ascending order and the corresponding orthonormalized eigenvectors $x_i(a^k)$, $i = 1, n$.
3) Check for convergence. If $\|\lambda(a^k) - \lambda^*\|$ is sufficiently small, stop.
4) Compute the Jacobian J and the vector b using Eqs. (3) and (4), respectively.
5) Calculate a^{k+1} by solving the system (5).
6) Set $k = k + 1$ and repeat from step 2.

Remarks

The method converges quadratically even in the case of specifications with repeated eigenvalues, provided the Jacobians computed are nonsingular.[5]

The method can be extended without difficulty to the case where $A(a)$ is nonlinearly related to a, say as

$$A(a) = A_0 + \sum_{i=1}^{n} g_i(a)\,A_i$$

where $g_i(a)$ are some known nonlinear functions of a.

Partial Generalized Inverse Eigenvalue Problem

Direct application of the previous method is not suitable for many problems in structural design. Frequently we are interested in a few low-order natural frequencies, and it is impractical to calculate all of the frequencies and mode shapes for large structures. In this section, we extend the method to the partial generalized inverse eigenvalue problem in which we specify only a few low-order frequencies. Also, we specify a set of lower limits for the design parameters. We write the basic equations in terms of the frequencies of the structure rather than the eigenvalues. This provides a natural scaling for the equations.

Definition of the Problem

Consider a structure with n elements and m displacement degrees of freedom. Let $a = (a_1, a_2, \ldots, a_n)$ be the set of design parameters. We write the stiffness matrix of the ith element of the structure in the form

$$g_i(a)\,k_i \tag{6a}$$

where $g_i(a)$ is a function of the design parameters and k_i is an $(m \times m)$ matrix independent of them. Similarly, we write the mass matrix of the ith element in the form

$$h_i(a)\,m_i \tag{6b}$$

where $h_i(a)$ is a function of the design parameters and m_i is an $(m \times m)$ matrix independent of them. Let m_0 be the constant $(m \times m)$ lumped mass matrix. We define the global stiffness matrix K and the global mass matrix M as

$$K = \sum_{i=1}^{n} g_i(a)k_i, \qquad M = m_0 + \sum_{i=1}^{n} h_i(a)\,m_i \tag{7}$$

Let $\omega^* = (\omega_1^*, \omega_2^*, \ldots, \omega_p^*)$ be a set of p positive real numbers arranged in ascending order. Define the set $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_p^*)$ using $\lambda_i^* = (2\pi\omega_i^*)^2$. The partial generalized inverse eigenvalue problem is to determine a such that λ^* is the set of the smallest p eigenvalues of the generalized eigenvalue problem $Kx = \lambda\,Mx$ and such that $a_i \geq l_i$, $i = 1, n$, where $l = (l_1, l_2, \ldots, l_n)$ is the vector of specified lower bounds for the design parameters.

In the case of a truss structure, Eq. (7) can be written in the following form where a represents the areas of cross section of the members:

$$K = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n, \qquad M = m_0 + a_1 m_1 + a_2 m_2 + \cdots + a_n m_n \tag{8}$$

Method

In the partial inverse eigenvalue problem, we have n design variables and we specify only p frequencies. This results in the following system of equations:

$$\omega_1(a) = \omega_1^*,\quad \omega_2(a) = \omega_2^*,\quad \ldots,\quad \omega_p(a) = \omega_p^* \tag{9}$$

This is a set of p equations in n unknowns, with $p < n$. The application of the Newton-Raphson method for Eq. (9) involves the solution of an underdetermined system of linear equations involving the rectangular Jacobian J in place of a fully determined system (2). Furthermore, we have specified a set of lower limits for the design parameters. So we modify the method accordingly.

Let $\lambda(a^k)$ be the set of the smallest p eigenvalues of the problem $Kx = \lambda Mx$ at the kth iterate $a = a^k$. Let $x_i(a^k)$, $i = 1, p$, be the eigenvectors corresponding to the smallest p eigenvalues. We assume that they are normalized with respect to M so that

$$x_i^T(a^k)\,M\,x_i(a^k) = 1 \qquad \text{for } i = 1, p$$

Then, as in Ref. 20, we can write

$$\frac{\partial \lambda_i}{\partial a_j} = x_i^T(a^k)\left[\frac{\partial K}{\partial a_j} - \lambda_i(a^k)\frac{\partial M}{\partial a_j}\right]x_i(a^k) \tag{10a}$$

For a truss problem, this expression, by virtue of Eq. (8) simplifies to

$$\frac{\partial \lambda_i}{\partial a_j} = x_i^T(a^k)[k_j - \lambda_i(a^k)m_j]x_i(a^k)$$

The (i,j)th element of the Jacobian of Eq. (9) is defined as

$$J_{ij} = \frac{\partial \omega_i}{\partial a_j} = \frac{1}{4\pi\sqrt{\lambda_i}} \frac{\partial \lambda_i}{\partial a_j} \tag{10b}$$

Applying the Newton-Raphson method, the next approximate solution $\boldsymbol{a}^{k+1}$ of Eq. (9) is determined as

$$\boldsymbol{a}^{k+1} = \boldsymbol{a}^k - \boldsymbol{d}$$

where the n vector $\boldsymbol{d}$ is a solution of the system

$$\boldsymbol{J}\,\boldsymbol{d} = \omega(\boldsymbol{a}^k) - \omega^* \tag{11}$$

This is an underdetermined system of p equations in n unknowns. In general, it has many solutions. The following method can be used to obtain a solution.

We partition the $(p \times n)$ matrix $\boldsymbol{J}$ into a square submatrix $\boldsymbol{J}_b$ of order p, consisting of p columns of $\boldsymbol{J}$ and a matrix $\boldsymbol{J}_f$ consisting of the remaining $n - p$ columns. This is done in such a way that $\boldsymbol{J}_b$ is nonsingular, and we call it a basis matrix. The vector $\boldsymbol{a}$ is similarly partitioned into a vector $\boldsymbol{a}_b$ of dimension p consisting of variables corresponding to the columns of the basis and a vector $\boldsymbol{a}_f$ consisting of the remaining variables. We call $\boldsymbol{a}_b$ the set of basic variables and $\boldsymbol{a}_f$ the set of fixed variables. We partition the vector $\boldsymbol{d}$ also into a vector $\boldsymbol{d}_b$ consisting of elements of $\boldsymbol{d}$ corresponding to the basic variables and a vector $\boldsymbol{d}_f$ consisting of the remaining elements. Now we write Eq. (11) as

$$[\boldsymbol{J}_b\ \boldsymbol{J}_f] \begin{bmatrix} \boldsymbol{d}_b \\ \boldsymbol{d}_f \end{bmatrix} = \omega(\boldsymbol{a}^k) - \omega^*$$

that is,

$$\boldsymbol{J}_b\,\boldsymbol{d}_b = -\boldsymbol{J}_f\,\boldsymbol{d}_f + \omega\,(\boldsymbol{a}^k) - \omega^* \tag{12}$$

By assigning arbitrary values for $\boldsymbol{d}_f$, we can solve Eq. (12) for $\boldsymbol{d}_b$, since $\boldsymbol{J}_b$ is nonsingular. We assign zero for $\boldsymbol{d}_f$ to get one solution. Then Eq. (12) becomes

$$\boldsymbol{J}_b\,\boldsymbol{d}_b = \omega(\boldsymbol{a}^k) - \omega^* \tag{13}$$

Fixing $\boldsymbol{d}_f$ at zero implies that, to compute the next approximation $\boldsymbol{a}^{k+1}$, we fix the variables $\boldsymbol{a}_f$ at their current values $\boldsymbol{a}_f^k$ and only the basic variables $\boldsymbol{a}_b$ are updated. The approximation $\boldsymbol{a}^{k+1}$ can be chosen as

$$\boldsymbol{a}_b^{k+1} = \boldsymbol{a}_b^k - \boldsymbol{d}_b$$

$$\boldsymbol{a}_f^{k+1} = \boldsymbol{a}_f^k$$

where $\boldsymbol{d}_b$ is obtained by solving Eq. (13).

Now there are two issues: 1) how to choose a nonsingular submatrix of $\boldsymbol{J}$ and 2) how to ensure that the new approximate solution does not violate the lower bound constraints.

One method to choose a nonsingular submatrix is to apply a lower and upper triangular matrix (LU) decomposition to the rectangular Jacobian. It has more columns than rows. So we opt to choose the pivots by column interchanges. In every step of the LU decompositon, we choose the pivot as the largest element from among the elements of the associated row. After p steps of the decomposition, we identify a nonsingular submatrix $\boldsymbol{J}_b$ made up of the pivot columns. The variables corresponding to these columns form the basic set. Also, this procedure automatically gives the LU factors of $\boldsymbol{J}_b$.

Note that the set of column indices that forms the basis may differ from one iteration to the next. And so do the basic variables. Thus it happens that all or many of the elements of $\boldsymbol{a}$ may get modified during the iterations, though in a single iteration we modify only p variables.

To maintain feasibility we take precaution at three stages:

1) The initial approximate solution $\boldsymbol{a}^0$ is chosen such that it satisfies the lower bound constraints.

2) We prevent a variable on its lower bound from entering the basic set. This is done during the LU decomposition by choosing pivots only from columns of $\boldsymbol{J}$ corresponding to variables strictly above the bound.

3) We implement Newton-Raphson's method using a direction of move and a step length. The direction of move is chosen as $\boldsymbol{d}_b$ itself. We determine the step length as the maximum value of the scalar α satisfying

$$\boldsymbol{a}_b^k - \alpha \boldsymbol{d}_b \geq \boldsymbol{l}_b \tag{15}$$

where $\boldsymbol{l}_b$ is a vector of dimension p having components of $\boldsymbol{l}$ corresponding to the basic variables. In case α exceeds 1.0, it is fixed at 1.0. Thus we modify Eq. (14) and calculate the next approximate solution using

$$\begin{aligned} \boldsymbol{a}_b^{k+1} &= \boldsymbol{a}_b^k - \alpha \boldsymbol{d}_b \\ \boldsymbol{a}_f^{k+1} &= \boldsymbol{a}_f^k \end{aligned} \tag{16}$$

In every iteration of Newton-Raphson's method, we need to compute the smallest p eigenvalues and the corresponding eigenvectors. The method of simultaneous iteration[21] is one of the suitable methods.

Algorithm Steps

The steps of the algorithm for the partial generalized inverse eigenvalue problem are the following:

1) Choose an initial feasible approximation $\boldsymbol{a}^0$ and set the iteration index k to 0.

2) Form $\boldsymbol{K}$ and $\boldsymbol{M}$ using Eq. (7) at $\boldsymbol{a} = \boldsymbol{a}^k$.

3) Compute the smallest p eigenvalues $\lambda_i(\boldsymbol{a}^k)$, $i = 1, p$, arranged in increasing order, of the problem $\boldsymbol{Kx} = \lambda\,\boldsymbol{Mx}$ and the corresponding eigenvectors $\boldsymbol{x}_i(\boldsymbol{a}^k)$, $i = 1, p$, normalized with respect to $\boldsymbol{M}$. Compute the smallest p frequencies $\omega_i(\boldsymbol{a}^k)$, $i = 1, p$.

4) Perform a convergence test. If $\|\omega(\boldsymbol{a}^k) - \omega^*\|$ is sufficiently small, stop.

5) Compute the Jacobian $\boldsymbol{J}$ using Eqs. (10a) and (10b).

6) Choose a $(p \times p)$ nonsingular submatrix $\boldsymbol{J}_b$ of $\boldsymbol{J}$ such that the current values of the variables corresponding to the columns of $\boldsymbol{J}_b$ are strictly above the lower bounds. Partition $\boldsymbol{a}$ into a basic set $\boldsymbol{a}_b$ consisting of variables corresponding to the columns of $\boldsymbol{J}_b$ and a fixed set $\boldsymbol{a}_f$ consisting of the remaining variables.

7) Determine the direction of move $\boldsymbol{d}_b$ by solving Eq. (13).

8) Determine the step length as the maximum value of α satisfying Eq. (15). If $\alpha > 1.0$, choose $\alpha = 1.0$.

9) Calculate the next approximation to the solution using Eq. (16).

10) Set $k = k + 1$ and repeat from step 2.

Remarks

Fix a variable on the bound only if it causes α in step 8 to become zero, leaving a wider choice of columns for the basis.

It is not necessary that we specify all of the smallest p frequencies. Instead we may specify p frequencies of desired orders. All that we need is a procedure to compute the eigenpairs of exactly these specified orders in step 3. The convergence check in step 4 involves only the frequencies of these specified orders. The order of $\boldsymbol{J}_b$ in step 6 is the number of the specified frequencies, and it need not be the order of the largest order frequency we have specified. However, when we use the method of simultaneous iteration, we may have to compute all of the eigenpairs up to the largest order specified and may select the desired ones from among them.

In many design problems, we may not be interested in specific values for the frequencies. Instead, we may specify certain bands that should not contain any frequency. In such cases, it is found that an interactive use of the preceding method is very useful. We start with some guess design, see the distribution of the frequencies, and then decide which frequency or frequencies should be increased or decreased. Frequencies can be moved one at a time or in groups. Thus, to solve one design problem, we may need a session initiating the inverse eigenvalue problem several times.

The algorithm, may fail to converge on three counts. One is that there may not exist a solution for the given specifications. We give an example for this in the next section. The second case is that the initial design may be far from the solution. In this case it may be possible to reach the solution in a step-by-step way. We solve a sequence of inverse eigenvalue problems starting with some specifications not far from the frequencies of the initial design. We can improve the design by trying to move the frequencies toward the desired specifications in convenient steps. Another obvious approach is to use trial and error using different initial designs.

The third reason is that we may run short of columns to select a basis from the Jacobian. This happens because the present method stipulates that once a variable reaches its bound, it is fixed at it permanently. The corresponding column of the Jacobian is not considered for the selection of the basis any further. This is a shortcoming of the method.

The proposed method provides no facility to minimize the weight of the structure or even to control the increase in the weight. This is another shortcoming, though one may exercise a certain amount of control by imposing upper bound constraints also for the design variables.

For a given specification, we may get different solutions from different starting points. This is because the solution in general is not unique.

Implementation of the method for large problems poses no special difficulty. The matrix J is dense and requires np words of storage. The storage requirement increases linearly with n. The LU decomposition of J needs $\mathcal{O}(np^2)$ operations, and this number also increases linearly with n. Usually p is small and these two resources do not become severe constraints.

For some types of structural elements it may not be possible to express an element stiffness or mass matrix directly in the form of Eqs. (6a) and (6b). However, it may be possible to express it as a linear or nonlinear combination of a few matrices of quantities independent of the design parameters. A typical example is the beam element in which the area of cross section and the moments of inertia are considered as functions of the design variables. In such cases also, the method presented here can be applied.

The number of design parameters need not be the same as the number of structural elements. We note, however, that the number of columns in the Jacobian is the same as the number of design parameters.

Computational Results

In this section we present some computational results to bring out the usefulness of the partial inverse eigenvalue problem in structural design. The first example is a 10-bar truss problem. Eight different designs of this structure are given corresponding to eight sets of frequency specifications. In the second example, we consider a 25-bar truss problem to illustrate how the method can be used interactively to design a structure for which a few bands, devoid of any frequency, are specified. Finally, a 100-bar truss problem is discussed to indicate that the method holds promise for use in the design of large structures of practical interest. In all of the problems discussed, the design variables are the area of cross section of the members and the specifications are in terms of frequencies, Hz.

Ten-Bar Truss

Figure 1 shows a truss structure with n bays. Each bay consists of five-bar elements. When $n = 2$, we have the 10-bar

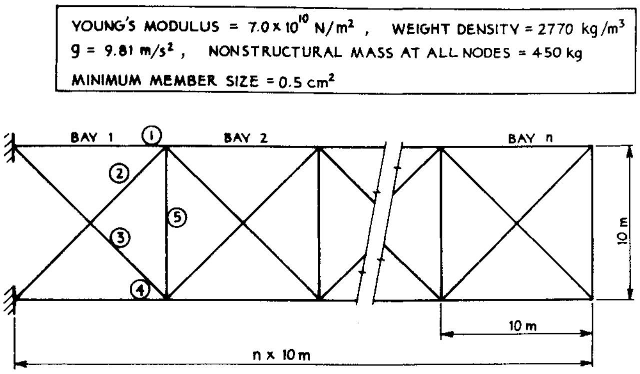

Fig. 1 Truss structure with n bays.

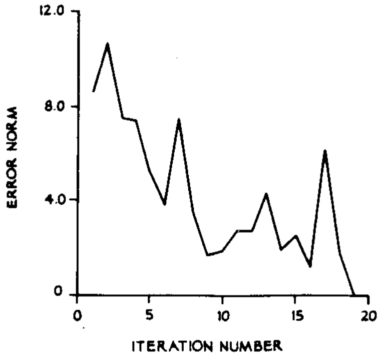

Fig. 2 Iteration history for specification 1 of the 10-bar truss.

Table 1 Frequency specifications for 10-bar truss

Spec. no.	Specifications, Hz			
1	$\omega_1 = 10$	$\omega_2 = 15$		
2	$\omega_1 = 5$	$\omega_5 = 30$		
3	$\omega_2 = 10$	$\omega_3 = 10$		
4	$\omega_1 = 5$	$\omega_3 = 15$	$\omega_5 = 25$	
5	$\omega_2 = 20$	$\omega_3 = 30$	$\omega_4 = 40$	
6	$\omega_1 = 10$	$\omega_3 = 30$	$\omega_5 = 50$	$\omega_7 = 70$
7	$\omega_1 = 5$	$\omega_2 = 10$	$\omega_3 = 15$	$\omega_4 = 20$
8	$\omega_1 = 5$	$\omega_2 = 10$	$\omega_3 = 15$	$\omega_4 = 20$
	$\omega_5 = 25$	$\omega_6 = 30$	$\omega_7 = 35$	$\omega_8 = 40$

Table 2 Designs for 10-bar truss satisfying specifications in Table 1

Spec.		Member sizes, cm²				
no.	Bays	1	2	3	4	5
1	1	76.36	78.99	16.07	47.56	0.50
	2	6.82	37.97	79.42	68.48	29.15
2	1	15.78	5.83	13.10	24.45	9.96
	2	7.30	10.17	5.00	5.00	11.76
3	1	6.53	5.00	4.65	5.60	5.00
	2	4.27	5.00	5.00	5.00	5.00
4	1	12.93	5.00	18.10	19.56	5.97
	2	7.83	11.73	5.00	8.14	8.12
5	1	40.93	5.78	100.70	5.00	55.40
	2	24.83	34.43	23.50	34.91	20.51
6	1	78.31	34.57	47.21	58.68	26.80
	2	42.39	43.35	34.74	27.53	32.63
7	1	18.31	18.66	6.17	14.18	5.00
	2	1.82	4.30	11.63	20.54	7.96
8	1	17.99	17.33	7.54	17.57	4.92
	2	1.75	4.31	11.69	12.03	8.76

Table 3 Frequencies of the 10-bar truss designs in Table 2

Spec.	Frequencies, Hz								Iterations	Weight, kg
1	10.00	15.00	27.45	33.95	51.19	58.10	73.65	88.27	20	1466.3
2	5.00	12.87	14.06	20.95	30.00	32.91	34.85	39.46	7	339.3
3	3.31	10.00	10.00	17.44	20.10	23.36	24.43	27.35	8	164.0
4	5.00	13.19	15.00	21.61	25.00	29.83	33.67	40.40	7	329.3
5	4.43	20.00	30.00	40.00	43.95	56.43	65.80	75.30	8	1147.1
6	10.00	27.53	30.00	47.92	50.00	59.53	70.00	79.86	9	1364.0
7	5.00	10.00	15.00	20.00	24.39	29.65	35.42	45.90	9	347.5
8	5.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00	8	334.7
Initial design	3.16	9.63	10.17	17.48	20.10	23.36	24.19	27.32		161.4

Table 4 Intermediate specifications and frequencies for 25-bar truss

	Least seven frequencies, Hz							Iterations	Weight, kg
Initial design	0.77	3.71	4.40	8.08	12.10	12.73	15.24	—	403.6
Specifications $\omega_1 = 2.0$	2.00	6.56	9.51	11.64	15.86	17.74	19.31	15	1349.6
$\omega_5 = 14.0$, $\omega_6 = 21.0$	1.98	6.84	9.57	11.29	14.00	21.00	21.05	12	1397.4
$\omega_1 = 2.0$, $\omega_5 = 14.0$, $\omega_6 = 21.0$	2.00	6.85	9.59	11.53	14.00	21.00	21.03	9	1410.1

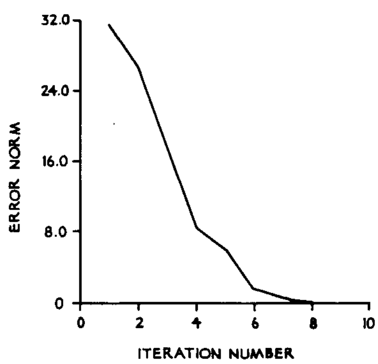

Fig. 3 Iteration history for specification 5 of the 10-bar truss.

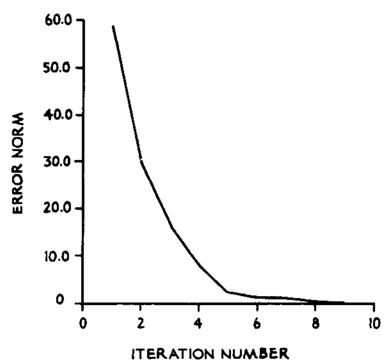

Fig. 4 Iteration history for specification 6 of the 10-bar truss.

truss problem. We give eight sets of specifications in Table 1. For specifications 1–7, an initial design of 5.0 cm^2 for each member is taken. The final design of specification 7 is used as the initial design for specification 8.

Table 2 gives the final designs obtained using the present method, satisfying the specifications in Table 1. Table 3 gives the frequency distributions of the final designs of Table 2 along with those of the initial design.

In specification 3, we give two repeated frequencies. No special difficulty in convergence was observed in this case. The initial designs have a reasonably well-separated frequency spectrum. This separation is more or less maintained for the final designs also. No tendency of clustering is noticed.

Figures 2, 3, and 4 give the plots of the square norm of the error against the iteration numbers, corresponding to specifications 1, 5, and 6, respectively. Figure 3 shows a smooth decrease in the error norm, whereas Fig. 4 shows an increase in the error norm once. The plots for specifications 2, 3, and 8, not presented here, are similar to Fig. 3. The plots for specifications 4 and 7 are similar to Fig. 4. Figure 2 shows convergence after violent oscillations, probably because specification 1 is relatively far from the initial frequencies.

Table 3 also gives the weight of the final design corresponding to each specification. It can be observed that the higher the frequencies, the greater the weight, generally. The two heaviest designs are those corresponding to the high specifications 6 and 1. In specification 3, the specified frequencies are close to the corresponding ones of the initial design, and the weight of the final design is close to that of the initial design.

All attempts at making the first two frequencies equal at some value between 4.0 and 50.0 Hz failed. This leads to the conclusion that there may not exist a solution to this specification.

Twenty-Five Bar Truss

When $n = 5$ in Fig. 1, we get a 25-bar truss. This problem illustrates the interactive use of the method. It also demonstrates how the method can be used for design problems with specified bands containing no frequencies.

The least seven frequencies of this structure, with an initial design of 5.0 cm^2 for each member, are given in Table 4. We seek a design for which there are no frequencies in the bands 0.0–2.0 and 15.0–20.0 Hz. We call these two bands the first band and the second band, respectively.

For the initial design, the first frequency is in the first band, and there is at least one in the second. Our first attempt is to increase the first frequency to the upper limit of the first band and so the first specification is $\omega_1 = 2.0$ Hz. The second row of Table 4 gives the frequency distribution of the new design. All of the frequencies are increased and at least three of them, viz., ω_5, ω_6, and ω_7 lie in the second band. Now we try to bring ω_5 below the lower limit of the second band and increase ω_6 beyond the upper limit. So the next set of specifications is $\omega_5 = 14.0$ Hz and $\omega_6 = 21.0$ Hz, and we try to improve on the design just obtained. Note that in this step we do not specify values for the frequencies ω_1–ω_4.

The frequency distribution of the new design for this specification is presented in the third row of Table 4. Note that the first frequency falls in the prohibited first band. Now we formulate our last specification as $\omega_1 = 2.0$ Hz, $\omega_5 = 14.0$ Hz, and $\omega_6 = 21.0$ Hz. One more initiation of the method starting from the latest design leads us to the final design given in Table 5. The last row of Table 4 contains the frequencies of the final design. There are now no frequencies in the specified bands. The interactive use of the method has helped us in using the knowledge of the frequency distribution in one step to formulate the specifications for the next.

Figures 5 and 6 give the plots of the square norm of the error for the first and third specifications, respectively. For the first, the decrease in the error norm is monotonic. In the third step, we gave a specification very close to the frequencies of the immediately preceding design. However, as shown in Fig. 6, the first iteration of this step took us to a point farther from the specification, and we missed the benefit of being very close.

Table 4 gives the weights of the intermediate designs. The first specification has caused an increase in all of the seven low-order frequencies, and the weight of the structure has also increased. For the second design, some of the frequencies decreased whereas some increased. The net effect is an increase in the weight. For the final design, almost all of the frequencies have marginally increased, resulting in a similar increase in the weight.

One-Hundred-Bar Truss

We get a 100-bar truss when there are 20 bays in Fig. 1. The purpose of this problem is to indicate that the method can be used for large problems as well. An initial design of 50.0 cm^2 for each member is used. The least seven frequencies of this design are presented in Table 6. The specification is $\omega_3 = 3.5$ Hz, $\omega_4 = 4.5$ Hz, and $\omega_5 = 6.5$ Hz. The design satisfying this specification is achieved in 11 iterations. Table 7 presents the final design. The frequency distribution of this design is given in Table 6. Figure 7 gives the plot for the convergence history.

Table 5 Final design for the 25-bar truss

	Member sizes, cm^2				
Bays	1	2	3	4	5
1	51.00	5.00	37.33	67.33	7.57
2	41.83	26.50	9.18	31.57	5.00
3	24.95	2.63	16.15	30.89	4.91
4	20.44	14.31	0.50	12.64	7.33
5	6.12	8.16	9.84	5.41	8.27

The last column of Table 6 shows that the weight of the structure has increased. Recall that all of the three specified frequencies are higher than the corresponding initial ones.

Tables 3, 4, and 6 give the number of iterations taken for convergence. It is observed that the number of iterations do not increase either with the size of the structure or with the number of frequency specifications. It may take more iterations if the initial frequencies are far from the specifications.

Further Work

In the present implementation of the method, we have opted to fix a variable forever on its lower bound if it happens to fall on it at some iteration. This is not mandatory. That variable may show a tendency to move into the feasible region at some subsequent iteration. If so, it can be released from the bound. The corresponding column of the Jacobian becomes a candidate for building a basis. We need a technique to determine *when* to release it. This might prevent the method from breaking down for lack of a basis.

There are two levels of degrees of freedom available in the method. One is the choice of a basis. There may exist many nonsingular submatrices in the Jacobian, and several of them can be good candidates to become the basis. The other is the choice of values for the quantity d_f in Eq. (12). Our choice of zero for it is arbitrary. There may be many admissible values. A judicious exercise of these two levels of degrees of freedom must enable us to determine the minimum weight design satisfying our specifications or at least to control the growth of

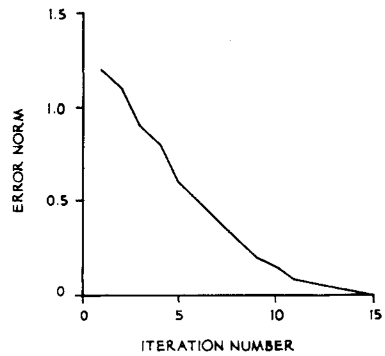

Fig. 5 Iteration history for the first specification of the 25-bar truss.

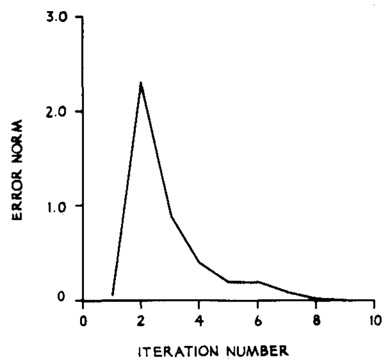

Fig. 6 Iteration history for the last specification of the 25-bar truss.

Table 6 Frequencies for 100-bar truss

	Least seven frequencies, Hz							Iterations	Weight,kg
Initial design	0.18	1.08	2.92	3.59	5.42	8.43	10.74		16,144.7
Final design	0.26	1.41	3.50	4.50	6.50	9.32	13.01	11	22,362.8

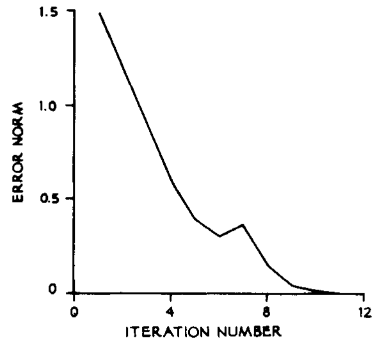

Fig. 7 Iteration history for the 100-bar truss.

Table 7 Final design for 100-bar truss

	Member sizes, cm^2				
Bays	1	2	3	4	5
1	270.03	50.00	50.00	421.07	50.00
2	227.61	50.00	50.00	105.55	50.00
3	317.16	50.00	50.00	211.83	50.00
4	164.85	50.00	50.00	185.78	50.00
5	153.28	50.00	50.00	50.00	50.00
6	136.36	50.00	50.00	50.00	50.00
7	73.60	50.00	50.00	50.00	50.00
8	50.00	50.00	50.00	43.36	50.00
9	47.39	50.00	50.00	50.00	50.00
10	50.00	50.00	50.00	50.00	50.00
11	120.73	50.00	50.00	187.92	50.00
12	50.00	50.00	50.00	50.00	50.00
13	50.00	50.00	50.00	50.00	50.00
14	50.34	50.00	50.00	50.00	50.00
15	50.00	50.00	50.00	131.43	50.00
16	118.17	50.00	50.00	50.00	50.00
17	115.13	50.00	50.00	164.40	50.00
18	48.80	50.00	50.00	50.00	50.00
19	50.00	50.00	50.00	50.00	50.00
20	50.00	50.00	50.00	50.00	50.00

weight. We can treat weight itself as another constraint. On the other hand, one might pose the whole problem as one of optimization where weight is the objective function and the specifications are the constraints, and the proposed method may serve to find an initial feasible design. Investigation along these lines should be of great utility.

There are several variants for the Newton-Raphson method.[22] Numerical experimentation to find out the one that provides faster and steady convergence for the present problem is another useful work. A few algorithmic options have been suggested for the ordinary inverse eigenvalue problem.[4] It must be possible to adapt them to increase the efficiency of the present method.

Conclusion

We propose a method by which a structural designer can exercise a certain amount of freedom in choosing the natural frequencies of a structure. This is derived as an extension of a method for the ordinary inverse eigenvalue problem. We have discussed a few examples to demonstrate the usefulness of the method in structural design. Though the method is illustrated for truss problems only, the approach is general for application to any type of structure.

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